

# Symplectic Geometry on Quantum Plane

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## Abstract

A study of symplectic forms associated with two dimensional quantum planes and the quantum sphere in a three dimensional orthogonal quantum plane is provided. The associated Hamiltonian vector fields and Poissonian algebraic relations are made explicit.

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Originated from the investigation of trigonometric and hyperbolic solutions of the quantum Yang-Baxter equation (QYBE)[1, 2], the quantum groups [3] have attracted much attention to mathematicians and physicists. This is because of their relations with many physical aspects such as exactly soluble models in statistical mechanics, conformal field theory, integrable model field theories [2, 4] and the fact that they have a rich mathematical structure, see e.g., [5]. In addition by defining a consistent differential calculus on the non-commutative space of the quantum groups [6], quantum groups supply concrete examples of non-commutative differential geometry [7].

The quantum plane is a space upon which the quantum group acts by linear transformations and whose coordinates belong to a non-commutative associative algebra [8]. By using the differential geometry method in [6] and interpreting the dual space of the quantum plane as differentials of the coordinates, covariant differential calculus on the quantum plane has been developed, so that the quantum plane provides a simple example for non-commutative differential geometry [9]. From the differential calculus on the three dimensional quantum plane, the quantum Schrödinger equation and the  $q$ -deformation of the hydrogen atom have been studied [10].  $q$ -deformed integrals on the quantum plane have also been studied, together with quantum versions of Cauchy's and Stokes' theorems [11].

In this paper we study symplectic geometry, Poisson algebraic structures and dynamics on quantum planes. We give explicit expressions of the symplectic forms associated with Hamiltonian vector fields and the corresponding Poisson algebraic relations for the two dimensional quantum plane and the quantum sphere in a three dimensional orthogonal quantum plane. It is found that for some quantum planes the symplectic structures exist for a special value of the deformation parameter  $q$ .

The quantum plane is defined in terms of  $n$  algebraic variables  $x^i$ ,  $i = 1, 2, \dots, n$ , building a noncommutative algebra  $A$  over  $\mathbb{C}$ , satisfying the commutation relations [8, 9],

$$x^i x^j = q x^j x^i, \quad i < j \quad (1)$$

where  $q$  is a complex number,  $q \neq 0$ . In addition one has algebraic variables  $\xi^i$ ,  $i =$

1, 2, ..., n, building a noncommutative algebra  $A^\partial$ , satisfying

$$\xi^i \xi^i = 0, \quad \xi^i \xi^j = -\frac{1}{q} \xi^j \xi^i, \quad i < j. \quad (2)$$

By definition the space  $GL_q(n)$  of “quantum matrices” consists of  $x \times n$  matrices  $M = (m_{ij})_{i,j=1,\dots,n}$  with  $m_{ij}$  (non-commutative) algebraic elements, acting on  $(x^1, \dots, x^n)$  resp.  $(\xi^1, \dots, \xi^n)$  commuting with the  $x^k$  resp.  $\xi^k$  and preserving the commutation relations (1) resp. (2). In [9] an interpretation of (1), (2) is given in terms of a “differential calculus”, covariant with respect to  $GL_q(n)$ .

More generally one can consider variables  $x^i, i = 1, \dots, n$ , belonging to a non-commutative algebra  $\mathcal{A}$ , satisfying the commutation relations (with the usual summation convention):

$$x^i x^j - B_{kl}^{ij} x^k x^l = 0, \quad (3)$$

or (in tensor product notation),  $(E_{12} - B_{12})x_1 x_2 = 0$ , where  $E$  is the  $n^2 \times n^2$  unit matrix,  $B$  is a  $n^2 \times n^2$ -matrix with entries in  $\mathbb{C}$ ,  $x_1$  and  $x_2$  are two representations of the “column vector”  $\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$ .

Let  $\mathcal{F}$  denote the ring of all functions of the  $x^i, i = 1, \dots, n, x^i \in \mathcal{A}$ , defined by formal power series in the  $x^i$ , with coefficients in  $\mathbb{C}$ . Let  $\mathcal{A}^\partial$  be a non commutative algebra, generated by  $\xi^i \equiv dx^i, i = 1, \dots, n$ , and we suppose that  $\mathcal{A}^\partial$  operates on  $\mathcal{F}$  in the sense that the operation  $f \rightarrow \xi^i f$  is well defined for any  $f \in \mathcal{F}$ . Let  $d$  be the “exterior differential” on  $\mathcal{F}$  defined by  $d = \xi^i \partial_i, \partial_i$  being the derivative with respect to  $x^i$  (defined on the ring  $\mathcal{F}$ ). For a consistent definition of “differential calculus” on the “B-plane” (3), one needs all the commutation relations among  $x^i, \xi^i$  and  $\partial_i$ . In general  $x^i, \xi^i$  and  $\partial_i$  are not commuting with each other but are required to have the following commutation relations:

$$x_1 \xi_2 = C_{12} \xi_1 x_2, \quad (E_{12} + D_{12}) \xi_1 \xi_2 = 0, \quad \partial_2 \partial_1 - \partial_2 \partial_1 F_{12} = 0, \quad (4)$$

where  $C, D$  and  $F$  are matrices with entries in  $\mathbb{C}$  to be determined. By requiring that  $d$  be nilpotent  $d^2 = 0$  and satisfy the Leibniz rule  $d(fg) = (df)g + f(dg), f, g \in \mathcal{F}$ , one has, as in [9]:

$$\partial_k x^i = \delta_k^i + C_{kl}^{ij} x^l \partial_j, \quad \partial_k \xi^i = D_{kl}^{ij} \xi^l \partial_j \quad (5)$$

and the following conditions for the matrices  $D$ ,  $C$ ,  $B$  and  $F$ :

$$\begin{aligned}
(E_{12} - B_{12})(E_{12} - C_{12}) &= 0, \\
(E_{12} - B_{12})C_{23}C_{12} &= C_{23}C_{12}(E_{23} - B_{23}), \\
D_{23}C_{12}C_{23} &= C_{12}C_{23}D_{12}, \quad D = C^{-1}, \\
(E_{12} - F_{12})(E_{12} - C_{12}) &= 0, \\
(E_{23} - F_{23})C_{12}C_{23} &= C_{12}C_{23}(E_{12} - F_{12}).
\end{aligned} \tag{6}$$

The solutions of (6) are given by the  $\check{R}$ -matrices satisfying the Yang-Baxter equation,

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \tag{7}$$

where  $\check{R}_{ij}$  denotes the matrix on the complex vector space  $v \otimes v \otimes v$ , acting as  $\check{R}$  on the  $i$ -th and the  $j$ -th components and as the identity on the other components, i.e.  $\check{R}_{12} = \check{R} \otimes E$ ,  $\check{R}_{23} = E \otimes \check{R}$ . The  $A_n$ -type  $\check{R}$  matrices satisfy

$$(\check{R} - \lambda_1)(\check{R} - \lambda_2) = 0, \tag{8}$$

with two different eigenvalues  $\lambda_1 = -q^{-1}$  and  $\lambda_2 = q$  [5]. The solutions of  $B$ ,  $C$ ,  $D$  and  $F$  in the system (6) associated with an  $A_n$ -type matrix are given by [9]:

$$B = q^{-1}\check{R}, \quad C = q\check{R}, \quad F = B, \quad D = C^{-1}. \tag{9}$$

The  $\check{R}$  matrices satisfying the braid Yang-Baxter equation have three different eigenvalues  $\lambda_i$ ,  $i = 1, 2, 3$ , associated with  $B_n$ ,  $C_n$  and  $D_n$  type solutions [5, 12], and one has:

$$(\check{R} - \lambda_1)(\check{R} - \lambda_2)(\check{R} - \lambda_3) = 0. \tag{10}$$

For the  $B_n$  type, one gets  $\lambda_0 = q^{-2n}$ ,  $\lambda_1 = -q^{-1}$ ,  $\lambda_2 = q$ . The solutions of  $B$ ,  $C$ ,  $D$  and  $F$  in the system of equations (6) related to  $B_n$ -type  $R$ -matrices are given by [10]

$$C = q\check{R}, \quad D = C^{-1}, \quad B = F = E - Q, \tag{11}$$

where  $Q = (\check{R} - \lambda_0)(\check{R} - \lambda_2)(\lambda_1 - \lambda_0)^{-1}(\lambda_1 - \lambda_2)^{-1}$ ,  $\lambda_1 \neq \lambda_0$ ,  $\lambda_1 \neq \lambda_2$ , is one of the three projection operators associated with  $\check{R}$ .

To investigate the symplectic structures of the B-quantum plane (3) we have to define a q-deformed symplectic form, an exterior product, an inner product and a Hamiltonian vector field in analogy with the corresponding objects of the theory of ordinary symplectic manifolds [13]. Let  $V_x$  (resp.  $V_x^*$ ) denote the vector space spanned by the basis  $\{\partial_i \equiv \frac{\partial}{\partial x^i}\}$  (resp.  $\{dx^i\}$ ),  $i = 1, \dots, n$ , at  $x \in \mathcal{A}$ , so that, with  $\langle, \rangle$  being the inner product between  $V_x$  and  $V_x^*$ :

$$\langle \partial_i, dx^j \rangle = \delta_i^j, \quad (12)$$

and for  $f \in \mathcal{F}$ ,

$$\langle f \partial_i, x^k dx^j \rangle = f C_{im}^{kj} x^m, \quad (13)$$

where  $C$  is given by (9) (resp. (11)) for the  $A_n$  (resp.  $B_n$ )-type case. Generally a vector in  $V_x$  (resp.  $V_x^*$ ) has the form  $\sum_{i=1}^n a_i(x) \partial_i$ ,  $a_i(x) \in \mathcal{F}$  (resp.  $\sum_{i=1}^n b_i(x) dx^i$ ,  $b_i(x) \in \mathcal{F}$ ). The general inner product of the form  $\langle f \partial_i, g dx^j \rangle$ ,  $f, g \in \mathcal{F}$ , can be deduced from (13), by using linearity.

Set  $V = \cup_x V_x$  (resp.  $V^* = \cup_x V_x^*$ ). We call  $f$  (resp.  $X^*$ ) a zero form (resp. one form) if  $f \in \mathcal{F}$  (resp.  $X^* \in V^*$ ). Let  $dx^i \otimes dx^j$  be an element in the tensor space of  $V_x^* \otimes V_x^*$ . Let  $\Gamma = (\Gamma_{lm}^{jk})$ ,  $j, k, l, m = 1, \dots, n$ , with entries in  $\mathbb{C}$ , be a solution of the Yang-Baxter equation (7). We define

$$dx^i \wedge dx^j = dx^i \otimes dx^j - \Gamma_{kl}^{ij} dx^k \otimes dx^l. \quad (14)$$

Since  $\Gamma$  is a solution of QYBE, the exterior algebra so defined is associative [15]. From the definition it follows

$$\langle \partial_i, dx^j \wedge dx^k \rangle = \delta_i^j dx^k - \delta_i^k dx^j. \quad (15)$$

Using (13) we then have

$$\langle \partial_i, x^j dx^k \wedge dx^l \rangle = C_{in}^{jk} x^n dx^l - C_{in}^{js} \Gamma_{st}^{kl} x^n dx^t. \quad (16)$$

From commutation relations as in (4), identifying  $\xi^i \xi^j$  with  $dx^i \wedge dx^j$ , and the definition of wedge product (14), we have that the matrix  $\Gamma$  must satisfy the following relation:

$$(D + E)(E - \Gamma) = 0. \quad (17)$$

A matrix  $\Gamma$  satisfying QYBE and (17) defines the exterior algebra on a quantum plane. With respect to the  $A_n$  case, from equation (8) and (17), we have  $\Gamma = \check{R}/q$  with  $\check{R}$  as in (8). Nevertheless, as  $\check{R}$  satisfies the cubic relation (10) for the case associated with  $B_n$  type algebras, a solution of  $\Gamma$  in terms of  $\check{R}$  is not obvious and not always possible.

Let  $M$  denote the quantum space defined by (3). We call a two form  $\omega$  on  $M$  closed if it satisfies

$$d\omega = 0. \quad (18)$$

Let  $\rfloor$  denote the left inner product defined by  $(X \rfloor \omega)(Y) = \omega(X, Y)$  for any two vectors  $X, Y$  and two form  $\omega$  on  $M$ . For any vector  $X \in V$ , if

$$X \rfloor \omega = 0 \quad \Rightarrow \quad X = 0, \quad (19)$$

we call the two form  $\omega$  non-degenerate. Condition (19) means that if  $\omega(X, Y) = 0$  for all  $Y \in V$ , then  $X = 0$ . We call a non-degenerate closed two form a symplectic form on  $M$ .

Let  $\omega$  be the symplectic form on  $M$ . For  $X \in V$ , if

$$X \rfloor \omega = -d f \quad (20)$$

for some  $f \in \mathcal{F}$ , we call  $X \equiv X_f$  the Hamiltonian vector field associated with  $f$ .

Let  $X_f, X_g$  be the Hamiltonian vector fields associated with  $f$  and  $g$  respectively,  $f, g \in \mathcal{F}$ . We define the Poisson bracket of  $f$  and  $g$  by

$$[f, g] = -X_f g. \quad (21)$$

For a given Hamiltonian  $H \in \mathcal{F}(M)$ , the corresponding dynamics is given by the following equation of motions:

$$\frac{dx^i}{dt} \equiv \dot{x}^i = [x^i, H]. \quad (22)$$

**Remark:** The formula (21) is formally the same as the one in symplectic geometry theory on usual manifolds or supermanifolds with  $U$ -numbers [14]. Nevertheless, in the quantum plane case, generally relations like  $[f, g] = \pm[g, f]$  for  $f, g \in \mathcal{F}$  do not hold.

We first investigate the symplectic structure on two dimensional quantum plane. In this case the matrix  $\check{R}$  is given by

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (23)$$

By (4), (5) and (9) we have all the commutation relations such as, with coordinates  $x$  and  $y$ ,

$$xy = qyx, \quad (24)$$

$$\xi^2 = \eta^2 = 0, \quad \xi\eta = -\frac{1}{q}\eta\xi, \quad (25)$$

where  $\xi = dx$ ,  $\eta = dy$ .

The symplectic form is

$$\omega = \xi \wedge \eta. \quad (26)$$

Explicitly from (14) we have

$$\omega = \xi \wedge \eta = q^{-2}\xi \otimes \eta - q^{-1}\eta \otimes \xi = -\eta \wedge \xi/q, \quad (27)$$

where  $\Gamma = \check{R}/q$  is used.

By formula (15) and (20) we have the Hamiltonian vectors  $X_x$  (resp.  $X_y$ ) associated with  $x$  and (resp.  $y$ ),

$$X_x = q\partial_y, \quad X_y = -q^2\partial_x. \quad (28)$$

Therefore the Poisson bracket of  $x$  and  $y$  is

$$[x, y] = -[y, x]/q = -q. \quad (29)$$

The dynamics on the quantum plane can be investigated using formula (22).

We now consider the quantum sphere in a three dimensional orthogonal quantum

plane. The corresponding  $\check{R}$ -matrix is

$$\check{R} = \begin{matrix} & \begin{matrix} ++ & +0 & +- & 0+ & 00 & 0- & -+ & -0 & -- \end{matrix} \\ \begin{matrix} ++ \\ +0 \\ +- \\ 0+ \\ 00 \\ 0- \\ -+ \\ -0 \\ -- \end{matrix} & \left( \begin{array}{cccccccc} q & & & & & & & \\ & d & & 1 & & & & \\ & & d(1-q^{-1}) & & -dq^{-1/2} & & q^{-1} & \\ & 1 & & 0 & & & & \\ & & -dq^{-1/2} & & 1 & & & \\ & & & & & d & & 1 \\ & & q^{-1} & & & & 0 & \\ & & & & & 1 & & 0 \\ & & & & & & & q \end{array} \right), \quad (30) \end{matrix}$$

where  $d = q - q^{-1}$  and the blank spaces mean that the corresponding entries are zeros. Let  $x^+$ ,  $x^0$  and  $x^-$  denote the coordinates,  $\xi^+ = dx^+$ ,  $\xi^0 = dx^0$  and  $\xi^- = dx^-$  (resp.  $\partial_+ = \frac{\partial}{\partial x^+}$ ,  $\partial_0 = \frac{\partial}{\partial x^0}$  and  $\partial_- = \frac{\partial}{\partial x^-}$ ) denote the differentials (resp. derivatives). From formulae (3) and (11) one has the relations:

$$x^+x^0 = qx^0x^+, \quad x^0x^- = qx^-x^0, \quad x^+x^- - x^-x^+ = (q^{-1/2} - q^{1/2})x^0x^0. \quad (31)$$

$$\begin{aligned} x^+\xi^+ &= q^2\xi^+x^+, & x^+\xi^0 &= q\xi^0x^+ + (q^2 - 1)\xi^+x^0, \\ x^+\xi^- &= \xi^-x^+ + (q^{-1} - q)q^{1/2}\xi^0x^0 - (q^{-1} - q)(q - 1)\xi^+x^-, \\ x^0\xi^+ &= q\xi^+x^0, & x^0\xi^0 &= q\xi^0x^0 + (q^{-1} - q)q^{1/2}\xi^+x^-, \\ x^0\xi^- &= q\xi^-x^0 + (q^2 - 1)\xi^0x^-, \\ x^-\xi^+ &= \xi^+x^-, & x^-\xi^0 &= q\xi^0x^-, & x^-\xi^- &= q^2\xi^-x^-. \end{aligned} \quad (32)$$

$$\begin{aligned} \partial_+x^+ &= 1 - (q^{-1} - q)(q - 1)x^-\partial_- + (q^2 - 1)x^0\partial_0 + q^2x^+\partial_+ \\ \partial_+x^0 &= (q^{-1/2} - q^{3/2})x^-\partial_0 + qx^0\partial_+, & \partial_+x^- &= x^-\partial_+ \\ \partial_0x^+ &= (q^{-1/2} - q^{3/2})x^0\partial_- + qx^+\partial_0, & \partial_0x^0 &= 1 + (q^2 - 1)x^-\partial_- + qx^0\partial_0, \\ \partial_0x^- &= qx^-\partial_0, & \partial_-x^+ &= x^+\partial_-, & \partial_-x^0 &= qx^0\partial_-, & \partial_-x^- &= 1 + q^2x^-\partial_-. \end{aligned} \quad (33)$$

The quantum sphere on plane (31) is defined by [16],

$$r^2 = q^{-1/2}x^+x^- + x^0x^0 + q^{1/2}x^-x^+, \quad (34)$$

where  $r^2 \in \mathbb{C}$  is the center of the algebra (31),  $r^2x^i = x^ir^2$ ,  $i = +, 0, -$ . By using relation (32) we have

$$x^0dx^0 + \sqrt{q}x^-dx^+ + \frac{1}{\sqrt{q}}x^+dx^- = 0. \quad (35)$$



When  $q = 1$  the relations (31), (32) and (33) become the ones of among the usual coordinates, differentials and derivatives, while the so defined quantum sphere (34) becomes the two dimensional sphere  $S^2$  in  $\mathbb{R}^3$ . Therefore (34) is a kind of quantum deformation of  $S^2$ , which results in the non-commutativity of coordinates. Another kind of deformation is the deformation from  $S^2$  to  $S_q^2$  in  $\mathbb{R}^3$ . The Poisson algebra on  $S_q^2$  is the  $q$ -deformed Lie algebra  $su_q(2)$  [17]. The latter deformation gives no problem concerning the existence of a corresponding symplectic structure for all values of  $q$ , as the Poisson algebraic structures on general two dimensional manifolds in  $\mathbb{R}^3$  can be explicitly given [18]. It might be interesting that similar to the fact that some usual manifolds, such as the three dimensional sphere, admit no symplectic structures, some quantum deformations like (34) may also have no symplectic structures for some values of  $q$ . We now give the symplectic structure on (34) for a special value of  $q$ .

From (11) we have

$$D = \begin{matrix} & \begin{matrix} ++ & +0 & +- & 0+ & 00 & 0- & -+ & -0 & -- \end{matrix} \\ \begin{matrix} ++ \\ +0 \\ +- \\ 0+ \\ 00 \\ 0- \\ -+ \\ -0 \\ -- \end{matrix} & \left( \begin{matrix} q^{-2} & & & & & & & & \\ & 0 & & q^{-1} & & & & & \\ & & 0 & & & & 1 & & \\ & q^{-1} & & -d/q & & & & & \\ & & & & q^{-1} & & d/\sqrt{q} & & \\ & & & & & 0 & & q^{-1} & \\ & & 1 & & d/\sqrt{q} & & a & & \\ & & & & & q^{-1} & & -d/q & \\ & & & & & & & & q^{-2} \end{matrix} \right), \end{matrix} \quad (36)$$

where  $a = d(1 - q^{-1})$ .

Besides the case  $q = 1$ , a solution of  $\Gamma$  satisfying QYBE and (17) exists when  $q = -1$ . In this case we have  $D^2 = I$  and  $\Gamma = D = \check{R}^{-1}$ . After some calculations we obtain the symplectic form,

$$\begin{aligned} \omega &= \frac{1}{ir^2}(-x^+ dx^- \wedge dx^0 + x^- dx^0 \wedge dx^+ - x^0 dx^+ \wedge dx^-) \\ &= \frac{1}{ir^2}(-x^+(dx^- \otimes dx^0 + dx^0 \otimes dx^-) + x^-(dx^0 \otimes dx^+ + dx^+ \otimes dx^0) \\ &\quad - x^0(dx^+ \otimes dx^- - dx^- \otimes dx^+)). \end{aligned} \quad (37)$$

One can check directly that  $d\omega = 0$ . The corresponding vector fields associated with  $x^{\pm,0}$

are:

$$X_{x^\pm} = -\left(x^\pm \partial_0 \pm i x^0 \partial_\mp\right), \quad X_{x^0} = -\left(x^- \partial_- + x^+ \partial_+\right). \quad (38)$$

By using relations (31), (33-35) it is straightforward to verify that (37) and (38) satisfy equation (20).

From formula (21) we have the Poisson relations among the  $x^{\pm,0}$ :

$$[x^\pm, x^\mp] = \pm i x^0, \quad [x^0, x^\pm] = x^\pm, \quad [x^\pm, x^0] = x^\pm. \quad (39)$$

We have studied the symplectic geometry, Poisson algebraic structures and dynamics on quantum planes. An interesting observation is that for some quantum planes, the symplectic structures may only exist for some special values of  $q$ . To investigate the general relations between Poisson algebraic structures and quantum planes, the geometrical quantizations, like in the case of usual manifolds [18], would be a challenging problem. It may be expected that, similar to the case of the quantum algebras discussed in [17, 19], the geometric quantization of quantum planes would show the difference in the roles played by the parameter  $q$  and the quantum Planck constant  $\hbar$ .

## References

- [1] C.N. Yang, Phys. Rev. Lett. **19**(1967)1312.
- [2] R.J. Baxter, *Exactly Solved Models in Statistical Physics*, Academic Press, New York 1982.
- [3] V.G. Drinfeld, Sov. Math. Dokl. 32(1985)254. Proc. I.C.M. Berkeley (1986)798-820.  
L.D. Faddeev, Leningrad Math. J. Vol.1 (1990), No.1.  
M. Jimbo, Lett. Math. Phys. 10(1985)63; Comm. Math. Phys. 102 (1986) 537.  
S. Woronowicz, Comm. Math. Phys. 111 (1987) 613.
- [4] E.K. Sklyanin, J. Soviet Math. 19(1982)1546.  
P.P. Kulish and E.K. Sklyanin, J. Soviet Math. 19(1982)1956.

- P.P. Kulish and N.Yu. Reshetikhin, J. Soviet Math. 23(1983)2435.
- H.J.de Vega, H. Eichenherr and J.M. Maillet, Nucl. Phys. B 240(1984)377.
- [5] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.
- Z.Q. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, World Scientific, 1993.
- C. Kassel, *Quantum Groups*, Springer-Verlag, New-York, 1995.
- Majid, S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.
- [6] S.L. Woronowicz, *Group Structure on Noncommutative Spaces, Fields and Geometry*, Singapore: World Scientific, 1986; Commun. Math. Phys. 111(1987)613; Commun. Math. Phys. 122(1989)125.
- [7] A. Connes, *Non-commutative differential Geometry*, Academic Press, San Diego, 1994.
- [8] Yu I Manin, *Quantum Groups and Non-commutative Geometry*, Les Publications du Centre de Recherches Mathématiques, Université de Montréal, CMR-1561, 1988; Commun. Math. Phys. 123(1989)163.
- [9] J. Wess and B. Zumino, Nucl. Phys. B (prol. Suppl.) 18B(1990)302-312.
- [10] X.C. Song and L. Liao, J. Phys. A 25(1992)623-634.
- [11] T. Brzezinski and J. Rembielinski, J. Phys. A 25(1992)1945-1952.
- [12] L. Faddeev, N. Reshetikhin and L. Takhtajan, Alg. Anal. 1(1987)178.
- M.L. Ge, L.Y. Wang, K. Xue and Y.S. Wu, Int. J. Mod. Phys. 4(1989)3351.
- [13] J. Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer Verlag, 1980.
- N. Woodhouse, *Geometric Quantization*, Oxford: Clarendon Press, 1980.
- B. Aebischer, M. Borer, M. Kälin, Ch. Leuenberger and H.M. Reimann, *Symplectic*

*Geometry*, Progress in Mathematics, Vol. 124, Birkhäuser 1994.

R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed. Addison-Wesley, Benjamin/Cummings, Reading, Mass.

[14] S.M. Fei, H.Y. Guo and Y. Yu, *Z. Phys. C* **45**(1989)339-344.

[15] A. El Hassouni, Y. Hassouni and M. Zakkari, *J. Math. Phys.* 35(1994)6096-6099.

[16] P. Podles, *Lett. Math. Phys.* 14(1987)193; 18(1989)107.

[17] S.M. Fei and H.Y. Guo, *J. Phys. A* 24(1991)1-10.

S.M. Fei, *J. Phys. A* 24(1991)5195-5214.

[18] S. Albeverio and S.M. Fei, *J. Phys. A* 31(1998)1211-1218.

[19] M. Flato and Z. Lu, *Lett. Math. Phys.* 21(1991)85.

M. Flato and D. Sternheimer, *Lett. Math. Phys.* 22(1991)155.